Identity.

https://www.linkedin.com/groups/8313943/8313943-6376814761484566533

Show that

$$\lfloor\sqrt{n}\,\rfloor+\lfloor\sqrt[3]{n}\,\rfloor+\ldots+\lfloor\lfloor\sqrt[n]{n}\,\rfloor=\lfloor\log_2 n\rfloor+\lfloor\log_3 n\rfloor+\ldots+\lfloor\log_n n\rfloor,\ \forall n\geq 2.$$

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Let $S_n := \left\{(a,b)\ \mid\ a,b\in\mathbb{N}\setminus\{1\} \text{ and } a^b\leq n\right\}$. Note that $a^b\leq n \Leftrightarrow a\leq\sqrt[b]{n} \Leftrightarrow a\leq[\sqrt[b]{n}\,]$

For any $b\in\{2,3,\ldots,n\}$ let $A_b := \left\{a\ \mid\ a\in\mathbb{N}\setminus\{1\} \text{ and } a\leq[\sqrt[b]{n}\,]\right\}$.

Then $S_n = \bigcup\limits_{b=2}^{n} A_b\times\{b\}$ and, therefore, $|S_n| = \sum\limits_{b=2}^{n}|A_b| = \sum\limits_{b=2}^{n}([\sqrt[b]{n}\,]-1) = \sum\limits_{b=2}^{n}[\sqrt[b]{n}\,]-(n-1)$.

Note that $a^b\leq n \Leftrightarrow b\leq\log_a n \Leftrightarrow b\leq[\log_a n]$

For any $a\in\{2,3,\ldots,n\}$ let $B_a := \left\{b\ \mid\ b\in\mathbb{N}\setminus\{1\} \text{ and } b\leq[\log_a n]\right\}$.

Then $S_n = \bigcup\limits_{b=2}^{n}\{a\}\times B_a$ and, therefore,

$$|S_n| = \sum_{a=2}^{n}|B_a| = \sum_{b=2}^{n}([\log_a n]-1) = \sum_{b=2}^{n}[[\log_a n]]-(n-1).$$

Thus, $\sum\limits_{b=2}^{n}[[\log_a n]]-(n-1) = \sum\limits_{b=2}^{n}[\sqrt[b]{n}\,]-(n-1) \Leftrightarrow \sum\limits_{a=2}^{n}[\log_a n] = \sum\limits_{b=2}^{n}[\sqrt[b]{n}\,]$.